Title: Note on semigroup homomorphisms

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NOTE ON SEMIGROUP HOMOMORPHISMS

Let A, B and C be semigroups written multiplicatively. Consider the direct product $A\times B$ of A and B and let

$$f : A\times B\to C$$

denote a homomorphism. M. Kuczma [1] has considered the possibility and uniqueness of a decomposition

$$f(x, y) = g(x)\, h(y) \tag{1}$$

where

$$g : A\to C \text{ and } h : B\to C \quad \text{are homomorphisms.} \tag{2}$$

He has proved some theorems under the assumption that the semigroups in question are commutative. In this note I shall prove that the commutativity can be replaced by a weaker condition of the commutativity type, namely by demanding the centres of A and B to be non-empty. Moreover, the case where A and B are monoids (i.e. semigroups with identity elements) will be considered (Theorem 3) and some examples of non-uniqueness of the decomposition (1) will be given.

The following theorems are generalized versions of the results of the note [1].

THEOREM 1. *Let A and B be semigroups with non-empty centres and let C be a group. If $f : A\times B\to C$ is a homomorphism, then it can be decomposed in the form (1), where the condition (2) is fulfilled. Moreover, this decomposition is unique.*

THEOREM 2. *Let A and B be semigroups with non-empty centres and let C be a semigroup which can be embedded in a group G. If $f : A\times B\to C$ is a homomorphism, then there are unique homomorphisms $g : A\to G$ and $h : B\to G$ such that (1) holds.*

COROLLARY. *Let A and B be semigroups with non-empty centres and let C be a semigroup embeddable in a group. If there is a homomorphism $f: A \times B \to C$ and if a decomposition (1) fulfilling (2) exists, then it is unique.*

In the case where the semigroups A, B and C are commutative we obtain the results of the note [1].

Theorem 2 and Corollary follow at once from Theorem 1, so I have to prove Theorem 1 only.

Proof of Theorem 1. Let a be an element of the centre of A and b an element of the centre of B. Then for any $x \in A$ and $y \in B$,

$$f(a, b)\, f(x, y) = f(ax, by) = f(xa, yb) = f(x, y)\, f(a, b),$$

i.e. $f(a, b)$ belongs to the centre of the image $Im\, f$ of the homomorphism f. Moreover, $(f(a, b))^{-1}$ commutes with all elements of $Im\, f$ (but need not belong to $Im\, f$), since from $f(a, b)\, f(x, y) = f(x, y)\, f(a, b)$ it follows

$$f(x, y)\, (f(a, b))^{-1} = (f(a, b))^{-1} f(x, y). \tag{3}$$

Now, we proceed as in [1]. We can define two mappings $g: A \to C$ and $h: B \to C$ by putting

$$g(x) = f(ax, b)\, (f(a, b))^{-1} \quad \text{and} \quad h(y) = f(a, by)\, (f(a, b))^{-1}. \tag{4}$$

Firstly, these are homomorphisms. Indeed, for $x, z \in A$ we have

$$f(a, b)\, g(xz)\, f(a, b) = f(a^2 xz, b^2) = f(axaz, b^2) = f(ax, b)\, f(az, b)$$

and using (3) and (4) we get $g(xz) = g(x)\, g(z)$. Similarly, h is a homomorphism.

Secondly, the decomposition (1) holds. In fact,

$$\begin{aligned} g(x)\, h(y) &= f(ax, b)\, (f(a, b))^{-1} f(a, by)\, (f(a, b))^{-1} \\ &= (f(a, b))^{-1} f(ax, b)\, f(a, by)\, (f(a, b))^{-1} \\ &= (f(a, b))^{-1} f(axa, b^2 y)\, (f(a, b))^{-1} \\ &= (f(a, b))^{-1} f(axa, byb)\, (f(a, b))^{-1} = f(x, y) \end{aligned}$$

Thirdly, this decomposition is unique. For suppose that $f(x, y) = g(x)\, h(y) = g_1(x) h_1(y)$ where $g_1: A \to C$ and $h_1: B \to C$ are homomorphisms. We see that $(g_1(x))^{-1} g(x) = h_1(y)\, (h(y))^{-1}$ for all x in A and y in B. Thus the functions on the left and right must be independent of x and y, respectively, i.e. there is a $c \in C$ such that $(g_1(x))^{-1} g(x) = c = h_1(y)\, (h(y))^{-1}$.

Hence $g(x) = g_1(x)\, c$ and $h_1(y) = c\, h(y)$.

Fix y and t in B. Then

$$ch(y)\, ch(t) = h_1(y)\, h_1(t) = h_1(yt) = ch(yt) = ch(y)\, h(t).$$

This can be cancelled to become $c = 1$. Thus $g = g_1$ and $h = h_1$ and the Theorem is proved.

The assumption about non-empty centres of semigroups A and B, appearing in both theorems, is naturally satisfied when the semigroups have two-sided identity elements, i.e. when they are *monoids*. The identity element of any monoid will be denoted by 1. If A and B are monoids and C is any semigroup and $f : A \times B \to C$ is a homomorphism, then a decomposition (1) fulfilling (2) does exist. For, $f(x, y) = f(x, 1)\, f(1, y)$ and we can take $g(x) = f(x, 1)$ and $h(y) = f(1, y)$. So, it remains the uniqueness of the decomposition to be considered. The assumptions about the semigroup C in both theorems above are rather strong, but they can hardly be weakened when we consider homomorphisms of $A \times B$ *into* C. It is clear that in order to prove our theorems we need only some regularity of the image $Im\, f$ and that it is not necessary to impose these conditions on the whole semigroup C. Having this in mind it is rather surprising that the followng theorem holds, where no assumptions on C are made.

THEOREM 3. *Let A and B be monoids and C a semigroup. If f is a homomorphism of $A \times B$ onto C then f can be uniquely decomposed in the form* (1), *where g and h satisfy* (2).

Proof. The existence of a decomposition has just been shown above. If $f(x, y) = G(x)\, H(y)$ where G and H are homomorphisms of A and B, respectively, into C, then we have

$$f(x, 1) = G(x)\, H(1) \text{ and } f(1, y) = G(1)\, H(y).$$

So, it will be sufficient to show that $H(1)$ is a right identity and $G(1)$ a left identity element of C. Let c be any element of C. Since f is a mapping of $A \times B$ *onto* C, there are $a \in A$ and $b \in B$ such that $c = f(a, b)$. Thus we have

$$cH(1) = f(a, b)\, H(1) = G(a)\, H(b)\, H(1) = G(a)\, H(b) = f(a, b) = c,$$

and similarly $G(1)c = c$. Thus $f(x, 1) = G(x)$ and $f(1, y) = H(y)$ and the Theorem is proved.

We end this note by showing some examples of non-uniqueness of the decomposition (1). Let A and B be arbitrary semigroups and let $C = \{0, 1\}$ with the usual multiplication as operation. Consider the zero homorphism $f : A \times B \to C$, i.e. $f(x, y) = 0$ for any $x \in A$ and $y \in B$. Next consider the zero homomorphisms $g_o : A \to C$ and $h_o : B \to C$ and a second couple of trivial homomorphisms $g_1 : A \to C$ and $h_1 : B \to C$ such that for any $x \in A$ and $y \in B$, $g_1(x) = 1$ and $h_1(y) = 1$. In such a case we have three different decompositions of the type (1):

$$f(x, y) = g_o(x)\, h_o(y) = g_1(x)\, h_o(y) = g_o(x)\, h_1(y).$$

Note that here f is not a mapping of $A \times B$ onto C.

In the second example A and B will be arbitrary semigroups and C will have to satisfy certain conditions: (*i*) C is a commutative semigroup, (*ii*) there are non-zero homomorphisms $g: A \to C$ and $h: B \to C$, (*iii*) there is an idempotent element e in C which is not identity. Consider the mapping $f: A \times B \to C$ where $f(x, y) = eg(x)\, h(y)$. This is clearly a homomorphism, for

$$f(x_1 x_2, y_1 y_2) = eg(x_1 x_2)\, h(y_1 y_2) = eg(x_1)\, h(y_1) \cdot eg(x_2)\, h(y_2) = \\ = f(x_1, y_1)\, f(x_2, y_2).$$

And now we have the following possibility for obtaining non-uniqueness of the decompositions (1):

$$f(x, y) = eg(x) \cdot h(y) = g(x) \cdot eh(y) = eg(x) \cdot eh(y).$$

I would like to thank Professor M. Kuczma for showing me his manuscript [1] before it has been published.

REFERENCE

[1] M. Kuczma: *Note on additive functions of several variables.* Prace Naukowe U. Śl., Prace Matematyczne 2 (1972), 49—51.

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NOTA O HOMOMORFIZMACH PÓŁGRUP

Streszczenie

Niech A, B, C będą półgrupami, $A \times B$ niech oznacza iloczyn prosty półgrup A i B. Udowodniono następujące twierdzenia:

TWIERDZENIE 1. *Jeśli każda z półgrup A i B ma niepuste centrum i C jest grupą, to każdy homomorfizm $f: A \times B \to C$ można w sposób jednoznaczny przedstawić w postaci: $f(x, y) = g(x)\, h(y)$, gdzie $g: A \to C$ oraz $h: B \to C$ są homomorfizmami.*

W przypadku gdy A, B, C są przemienne otrzymujemy stąd główny rezultat pracy [1].

TWIERDZENIE 3. *Jeśli A i B są monoidami i f jest homomorfizmem $A \times B$ na półgrupę C, to f można w sposób jednoznaczny przedstawić w postaci z twierdzenia 1.*

Wskazane zostały także przykłady niejednoznacznego rozkładu homomorfizmu f w rozważanej postaci.

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